The arithmetic rank of residual intersections and determinantal nullcones

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Abstract

The arithmetic rank of an ideal in a polynomial ring over an algebraically closed field is the smallest number of equations needed to define its vanishing locus set-theoretically. We determine the arithmetic rank of the generic m-residual intersection of a complete intersection ideal generated by t indeterminates for all $m \geq t$ and in every characteristic. We further compute the arithmetic rank of nullcone ideals arising from the classical actions of the special linear group, the orthogonal group, and the symplectic group, in every characteristic other than two.

Preliminaries

Let $\mathbb{K}$ be an algebraically closed field and I be an ideal in a polynomial ring S.

Definition: The *arithmetic rank* of I is the smallest number of elements needed to generate it up to radical

$$\operatorname{ara}(I) := \min\{k : \text{there exist } f_1, \ldots, f_k \in S \text{ with } \sqrt{f_1, \ldots, f_k} = \sqrt{I}\}.$$

Example: [Bar95] Let X be a 4×4 symmetric matrix of indeterminates and $S := \mathbb{K}[X]$. Then,

$$\operatorname{ara}(I_2(X)) = \begin{cases} 5 & \text{if } \operatorname{char}(\mathbb{K}) = 2 \\ 8 & \text{otherwise.} \end{cases}$$

Definition: Let $m \geq t \geq 1$ be integers and let $\mathbb{K}$ be any field or the integers. Let X and $\underline{y}$ be $m \times t$ and $t \times 1$ matrices of indeterminates respectively over $\mathbb{K}$. Let $R := \mathbb{K}[\underline{y}]$ and $S := R[X]$. Consider the *complete intersection* ideal $(\underline{y}) \subseteq R$ generated by the regular sequence $\underline{y} = y_1, \ldots, y_t$ in R. Let $\mathfrak{a} \subseteq S$ be the ideal generated by the entries of the matrix $X\underline{y}$. The ideal $\mathrm{RI}(m, \underline{y}) := \mathfrak{a} : (\underline{y})S$ is called the *generic m-residual intersection* of the *complete intersection* ideal $(\underline{y})$. By [HU90] we have that,

$$\mathrm{RI}(m, \underline{y}) = I_t(X) + I_1(X\underline{y}).$$

- The notion of residual intersections, introduced by Artin and Nagata [AN72], is the higher codimension analogue of links of ideals [PS74].
- The residual intersections of complete intersections arise as the defining ideals of certain varieties of complexes, introduced by Buchsbaum and Eisenbud [BE75].

Nullcone ideal: Consider a polynomial ring S over a field $\mathbb{K}$, and a group G acting on S via degree-preserving $\mathbb{K}$-algebra automorphisms. By the *nullcone ideal* of the action, we mean the expansion of the homogeneous maximal ideal of the invariant ring S^G to the polynomial ring S.

Determinantal nullcones: Let $t, m, n \geq 1$ be integers and $\mathbb{K}$ be an infinite field. Consider the action of a group $G \leq \mathrm{GL}_t(\mathbb{K})$ on the polynomial ring $S := \mathbb{K}[X, Y]$ via degree-preserving $\mathbb{K}$-algebra automorphisms, where $M \in G$ acts via

$$M : \begin{cases} X \mapsto XM^{-1} \\ Y \mapsto MY. \end{cases}$$

From [DCP76], the corresponding nullcone ideals are given below.

	G	S	Nullcone ideal
Special linear group	$\mathrm{SL}_t(\mathbb{K})$	$\mathbb{K}[X_{m\times t}, Y_{t\times n}]$	$\mathfrak{A} := I_t(X) + I_1(XY) + I_t(Y)$
Orthogonal group	$\mathrm{O}_t(\mathbb{K})$	$\mathbb{K}[X_{m\times t}, Y_{t\times n}]$	$\mathfrak{B} := I_1(XX^{\mathrm{tr}}) + I_1(Y^{\mathrm{tr}}Y) + I_1(XY)$
Symplectic group	$\mathrm{Sp}_{2t}(\mathbb{K})$	$\mathbb{K}[X_{m\times 2t}, Y_{2t\times n}]$	$\mathfrak{C} := I_1(X\Omega X^{\mathrm{tr}}) + I_1(Y^{\mathrm{tr}}\Omega Y) + I_1(XY)$

Note that $\mathrm{RI}(m, \underline{y})$ is the nullcone ideal $\mathfrak{A}$, when $n = 1$.

Results

Theorem [BMSMP26]: Let $m \geq t \geq 1$ be integers and let $\mathbb{K}$ be any field or integers. Let X and $\underline{y}$ be $m \times t$ and $t \times 1$ matrices of indeterminates, respectively. The arithmetic rank of the generic m-residual intersection ideal $\mathrm{RI}(m, \underline{y}) = I_t(X) + I_1(X\underline{y})$ in the polynomial ring $\mathbb{K}[X, \underline{y}]$ is

$$\operatorname{ara}(\mathrm{RI}(m, \underline{y})) = \begin{cases} t(m-t+1)+1 & t \geq 2, \\ m & t = 1. \end{cases}$$

Theorem [BMMS26]: Let $\operatorname{char}(\mathbb{K}) \neq 2$. Let m, n, t be positive integers with $t \leq \max\{m, n\}$. Let X and Y be matrices of indeterminates of sizes $m \times t$ and $t \times n$, respectively. Consider the nullcone ideal $\mathfrak{A} := I_t(X) + I_1(XY) + I_t(Y)$ from the action of $\mathrm{SL}_t(\mathbb{K})$ on $\mathbb{K}[X, Y]$. Then,

$$\operatorname{ara}(\mathfrak{A}) = t(m+n-t)+1.$$

Theorem [JPSW25, BMMS26]: Let $\operatorname{char}(\mathbb{K}) \neq 2$. Let X and Y be matrices of indeterminates of sizes $m \times t$ and $t \times n$, respectively. Consider the nullcone ideal $\mathfrak{B} := I_1(XX^{\mathrm{tr}}) + I_1(Y^{\mathrm{tr}}Y) + I_1(XY)$ from the action of $\mathrm{O}_t(\mathbb{K})$ on $\mathbb{K}[X, Y]$. Then,

$$\operatorname{ara}(\mathfrak{B}) = \binom{n+m+1}{2} - \binom{n+m+1-t}{2}.$$

Theorem [JPSW25, BMMS26]: Let $\operatorname{char}(\mathbb{K}) \neq 2$. Let X and Y be matrices of indeterminates of sizes $m \times 2t$ and $2t \times n$, respectively. Consider $\mathfrak{C} := I_1(X\Omega X^{\mathrm{tr}}) + I_1(Y^{\mathrm{tr}}\Omega Y) + I_1(XY)$, the nullcone ideal from the action of $\mathrm{Sp}_{2t}(\mathbb{K})$ on $\mathbb{K}[X, Y]$. Then,

$$\operatorname{ara}(\mathfrak{C}) = \binom{n+m}{2} - \binom{n+m-2t}{2}.$$

Algebra with a Straightening Law

Let $m \geq t \geq 1$ be integers. Let X and $\underline{y}$ be $m \times t$ and $t \times 1$ matrices of indeterminates respectively over a field $\mathbb{K}$. Let Q_i to be the i-th entry of the matrix $X\underline{y}$ and $[i_1, \ldots, i_n]$ to be the size n-minor of X with rows $i_1 < i_2 < \cdots < i_n$. Let $B := \{Q_1, \ldots, Q_m\} \cup \{[i_1, \ldots, i_n] \mid 1 \leq i_1 < \cdots < i_n \leq m\}$ be the set of $\mathbb{K}$-algebra generators of $\mathbb{K}[B]$. We define a partial order $<$ on B as follows:

- $Q_i \leq Q_j$ if $i \leq j$.
- $Q_j \leq [i_1, i_2, \ldots, i_n]$ if $j \leq i_n$.
- $[i_1, \ldots, i_n] \leq [j_1, \ldots, j_n]$ if $i_k \leq j_k$ for all k.

Theorem [BMSMP26]: The ring $\mathbb{K}[B]$ has dimension $d := t(m-t+1)+1$ with a homogeneous system of parameters given by

$$\left\{ \sum_{\operatorname{rank}(\mu)=i} \mu \ : \ i = 1, \ldots, t(m-t+1)+1 \right\}.$$

Example: Let X and $(\underline{y})$ be the 4×2 and 2×1 matrices of indeterminates, respectively. Then we have $\dim(\mathbb{K}[B]) = 7$; the polynomials

$$Q_1,\ Q_2,\ Q_3 + [1,2],\ Q_4 + [1,3],\ [1,4] + [2,3],\ [2,4], \text{ and } [3,4]$$

form a system of parameters of $\mathbb{K}[B]$. Equivalently, they realize the arithmetic rank of the ideal $\mathrm{RI}(4, (y_1, y_2))$.

Invariant Theory and Local Cohomology

Recall that $\mathrm{RI}(m, (\underline{y}))$ is the nullcone ideal $\mathfrak{A} = I_t(X) + I_1(X\underline{y})$. The above mentioned poset B consists of the minimal generators of $\mathfrak{A}$. When $\mathbb{K}$ is infinite, the *First Fundamental Theorem* for $\mathrm{SL}_t(K)$ states that the invariant ring is precisely the $\mathbb{K}$-algebra generated by the set B, i.e., $S^{\mathrm{SL}_n(K)} = \mathbb{K}[B]$, see [DCP76, Theorem 3.3].

Cohomological dimension: $\operatorname{cd}(I, S) := \max\{t \mid H_I^t(S) \neq 0\}$.

Local cohomology obstruction [JPSW25, Theorem 1.1]: Let $S := \mathbb{K}[X, \underline{y}]$ be a polynomial ring over a field $\mathbb{K}$ of characteristic zero, and consider $G := \mathrm{SL}_t(\mathbb{K})$, a linearly reductive group, acting on S by degree-preserving $\mathbb{K}$-algebra automorphisms. Let $T := S^G$ denote the ring of invariants, and m_T be the homogeneous maximal ideal of T. Then the nullcone ideal $m_T S$ has arithmetic rank $\dim(T)$.

Proof sketch: Here, $T = \mathbb{K}[B]$. Note that $\operatorname{cd}(\mathrm{RI}(m, \underline{y}), S) \leq \operatorname{ara}(\mathrm{RI}(m, \underline{y})) \leq d$. Since G is linearly reductive, the inclusion of $T \hookrightarrow S$ of T-modules splits. It follows that,

$$\mathrm{H}^d_{m_T S}(S) = \mathrm{H}^d_{m_T}(S) = \mathrm{H}^d_{m_T}(T) \otimes_T S \neq 0.$$

Positive characteristics: By [Hun81], $S/\mathrm{RI}(m, \underline{y})$ is a CM domain, since $\mathrm{RI}(m, \underline{y})$ defines a variety of complexes. Hence by [PS73], $\operatorname{ht}(\mathrm{RI}(m, \underline{y})) = \operatorname{cd}(\mathrm{RI}(m, \underline{y}), S) = m < d$.

Vanishing of Singular and Étale cohomology

Let $\mathbb{K}$ be algebraically closed. We follow the general strategy of Bruns and Schwänzl [BS90]. Let

- $U := (\mathbb{K}^{mt+t}) \setminus V(\mathrm{RI}(m, \underline{y}))$, $U_1 := (\mathbb{K}^{mt+t}) \setminus V(X\underline{y})$ and $U_2 := (\mathbb{K}^{mt+t}) \setminus V(I_t(X))$.
- Note that $U = U_1 \cup U_2$ and let $U_{12} := U_1 \cap U_2$.

The results follow from affine vanishing and the Mayer–Vietoris sequence for singular and étale cohomologies [Mil80, III.2.24].

Affine Vanishing Theorem:

(1) Let $\operatorname{char}(\mathbb{K}) = 0$. If X is a smooth complex variety of algebraic dimension $\mathfrak{D}$ that admits an open cover by k affines, then

$$\mathrm{H}^i_{\mathrm{sing}}(X, \mathbb{Q}) = 0 \quad \text{for all } i > \mathfrak{D} + k - 1.$$

(2) Let $\operatorname{char}(\mathbb{K}) = p > 0$. If X is a smooth variety of algebraic dimension $\mathfrak{D}$, over an algebraically closed field $\mathbb{K}$, that admits an open cover by k affines, then for any q invertible in $\mathbb{K}$

$$\mathrm{H}^i_{\text{ét}}(X, \mathbb{Z}/q\mathbb{Z}) = 0 \quad \text{for all } i > \mathfrak{D} + k - 1.$$

Mayer–Vietoris sequence:

$$\cdots \to \mathrm{H}^i_{\mathrm{sing}}(U, \mathbb{Q}) \to \mathrm{H}^i_{\mathrm{sing}}(U_1, \mathbb{Q}) \oplus \mathrm{H}^i_{\mathrm{sing}}(U_2, \mathbb{Q}) \to \mathrm{H}^i_{\mathrm{sing}}(U_{12}, \mathbb{Q}) \to \mathrm{H}^{i+1}_{\mathrm{sing}}(U, \mathbb{Q}) \to \cdots$$

Theorem [BMSMP26]:

(1) Let $\mathbb{K} = \mathbb{C}$ and $\mathfrak{D} := mt + t$, the algebraic dimension of U. Then

$$\mathrm{H}^{\mathfrak{D}+(mt-t^2+t)}_{\mathrm{sing}}(U, \mathbb{Q}) = \mathbb{Q}.$$

(2) Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$ and let m, t, $\mathfrak{D}$ and U be defined as above. Then

$$\mathrm{H}^{\mathfrak{D}+(mt-t^2+t)}_{\text{ét}}(U, \mathbb{Z}/q\mathbb{Z}) = \mathbb{Z}/q\mathbb{Z},$$

where q is a prime integer other than p.

We have that $\mathfrak{D} + (mt - t^2 + t) \leq \mathfrak{D} + \operatorname{ara}(\mathrm{RI}(m, \underline{y})) - 1$ and hence, $d \leq \mathrm{RI}(m, \underline{y})$.

Acknowledgments: I would like to thank the organizers for giving me this opportunity to present our work.

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